THERMAL WAVES IN THE GROUND UNDER INSULATION OF COOLERS

(TEMPERATURNYE VOLNY V GRUNTE POD IZOLIATSIEI KHOLODIL'NIKOV)

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At the present time, the calculation of heat transfer is based, in its most essential part, on the artificial introduction of a coefficient of heat transfer, α. With its aid, several phenomena occurring at a surface of contact between two materials are taken into account in a conventional way. The mean value of this coefficient is decisive for approximate engineering calculations and examples because it accounts in some way for the integral effect of the contact surface.

In mathematical terms, the complex nature of surfaces in thermal contact is translated into a boundary condition of the third kind with a coefficient of proportionality α. In many cases this boundary condition is purely local; it represents, in essence, the ratio of the normal derivative to the temperature difference at a point on the boundary of the contact surface. Generally speaking, it depends on the relevant physical parameters of the arrangement. This circumstance is well known and suggests the usefulness of providing mathematical solutions to a number of specific problems involving the above boundary conditions.

In addition, in the problems involving contact which have been considered by the author [1,11], the introduction of boundary conditions of the third kind proves to be possible, because in the general expression for the resistance to heat transfer, the component α^{-1} turns out to be small compared with the resistance of the insulating material.

The above statement, formulated in relation to thermal waves which may exist under the insulation of a cooler, does not constitute a particular case if only because this class of problems includes the calculation of the foundations of a large number of structures as well as problems involving diffusion and thermal diffusion. The essential feature of the analysis consists in the fact that it introduces new characteristic parameters in the form of generalized complex coefficients of heat transfer which express the non-steady nature of the phenomena under consideration.

Similar problems are of interest also in the field of acoustics and applied electrodynamics involving impedance-type boundaries. The difference between the present and latter problems consists in the fact that the numerical values of the appropriate dimensionless parameters are different, and this makes it possible to apply entirely different methods of approximation and to construct an exact solution of the problem with the aid of Mathieu's function.

1. Statement of problem. The temperature field in the ground under the insulation of a cooler consists of two components: (1) a steady temperature field determined by the mean values of the temperature of the air above the surface, of the water in the ground, and of the chambers of the cooler [1]; and (2) a non-steady field which arises in connection with fluctuation of the preceding temperatures about the respective mean values.

In order to analyze the non-steady field, we shall consider a cooler without a basement but with an insulated floor whose width is $2l$ and with an insulation of thickness δ. Let the functions $\theta^0(x, y, t)$ and $\theta(x, y, t)$ determine the non-steady temperature field in the insulation and in the ground, respectively. H denotes the depth of the ground water; λ_u and $\lambda_з$ denote the thermal conductivities of the insulation and the ground, respectively; α_c denotes the coefficient of heat transfer of the floor of the cooler; a_u and $a_з$ denote the thermal diffusivities of the insulation and the ground, respectively. The temperature field in a homogeneous ground is then given by Fourier's equation

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=\frac{1}{a_з}\frac{\partial\theta}{\partial t}\quad\text{for } -H\leqslant y\leqslant 0 \tag{1.1}$$

The boundary conditions are

$$\theta=\Delta\theta_0\quad\text{at } y=0,\ |x|>l \tag{1.2}$$

$$\lambda_з\frac{\partial\theta}{\partial y}=\lambda_u\frac{\partial\theta^\circ}{\partial y},\qquad \theta=\theta^\circ\quad\text{at } y=0,\ |x|<l \tag{1.3}$$

$$\theta=\Delta\theta_b\quad\text{at } y=-H \tag{1.4}$$

where $\Delta\theta_0(t)$ and $\Delta\theta_b(t)$ denote the departures of the temperatures of the air above the surface of the ground and underground water from their mean values θ_0 and θ_b.

The function $\theta^0(x, y, t)$ is determined by the condition

$$-\lambda_u \frac{\partial \theta^\circ}{\partial y} = \alpha_c (\theta^\circ - \Delta\theta_c) \quad \text{at} \quad y = \delta, \ |x| < l \tag{1.5}$$

where $\Delta\theta_c(x, t)$ denotes the deviation of temperature in the chambers of the cooler from the mean value $\theta_c(x)$.

In the case of a homogeneous insulation, the function $\theta^0(x, y, t)$ satisfies Fourier's equation (1.1) for $0 < y < \delta$ and $|x| < l$, it only being necessary to replace a_z by a_u. From an analysis of dimensions [1] it can be shown that with an accuracy which is sufficient for practical applications, the equation valid within the thickness of the insulation can be simplified to

$$\frac{\partial^2 \theta^\circ}{\partial y^2} = \frac{1}{a_u} \frac{\partial \theta^\circ}{\partial t} \quad \text{for } 0 \leqslant y \leqslant \delta, \qquad |x| < l \tag{1.6}$$

because the thickness of the insulation δ is much smaller than the width $2\ l$.

In what follows, we shall study the response of the system to a harmonic input, i.e. we shall assume

$$\begin{gathered} \Delta\theta_0 = Ae^{j\omega t}, \qquad \Delta\theta_b = Be^{j\omega t}, \qquad \Delta\theta_c = Ce^{j\omega t} \\ \theta(x, y, t) = \theta(x, y)\, e^{j\omega t}, \qquad \theta^\circ(x, y, t) = \theta^\circ(x, y)\, e^{j\omega t} \qquad (j = \sqrt{-1}) \end{gathered} \tag{1.7}$$

Here, and in what follows, it is necessary to consider only the real part of complex expressions which contain the exponential time-dependent term $\exp j\omega t$.

In view of (1.7), Equations (1.1) and (1.6) transform to

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = (1 + j)^2 \sigma_z{}^2 \theta \quad \text{for } -H \leqslant y \leqslant 0 \qquad \left(\sigma_z = \left(\frac{\omega}{2a_z}\right)^{1/2}\right) \tag{1.8}$$

$$\frac{\partial^2 \theta^\circ}{\partial y^2} = (1 + j)^2 \sigma_u{}^2 \theta^\circ \quad \text{for } 0 \leqslant y \leqslant \delta, \ |x| < l \qquad \left(\sigma_u = \left(\frac{\omega}{2a_u}\right)^{1/2}\right) \tag{1.9}$$

Equation (1.9) gives (1.10)

$$\theta^\circ(x, y) = C_1(x) \cosh \frac{\mu_u}{\delta} y + C_2(x) \sinh \frac{\mu_u}{\delta} y \ \text{ for } 0 \leqslant y \leqslant \delta \quad |x| < l \ (\mu_u = (1 + j)\, \sigma_u \delta)$$

where C_1 and C_2 denote constants of integration which depend on x.

Making use of relations (1.3), (1.5), (1.7) and (1.10) we obtain the following condition for $\theta(x, y)$:

$$-\lambda_2 \frac{\partial\theta}{\partial y} = k_1\theta - k_2 C \quad \text{at} \quad y = 0, \quad |x| < l \tag{1.11}$$

where k_1 and k_2 denote generalized complex coefficients of heat transfer which are determined by the relations

$$k_1 = \frac{\lambda_u \mu_u (\alpha_c \delta \cosh \mu_u + \lambda_u \mu_u \sinh \mu_u)}{\delta (\alpha_c \delta \sinh \mu_u + \lambda_u \mu_u \cosh \mu_u)}, \qquad k_2 = \frac{\lambda_u \mu_u \alpha_c}{\alpha_c \delta \sinh \mu_u + \lambda_u \mu_u \cosh \mu_u} \tag{1.12}$$

In the case of non-homogeneous insulation consisting of separate layers, the boundary condition (1.11) retains its form but the parameters k_1 and k_2 must be calculated with the aid of more complicated expressions. When $|\mu_u|$ is small ($\sinh \mu_u \approx \mu_u$, $\cosh \mu_u \approx 1$), the generalized complex coefficients of heat transfer assume the same steady-state value

$$k = \left(\frac{1}{\alpha_c} + \frac{\delta}{\lambda_u}\right)^{-1}$$

However, in the present case, the quantity $|\mu_u|$ is not small. In fact, for an insulation consisting of glass wool $a_u = 10^{-3}$ m^2/hr and hence, for annual fluctuations $\mu_u = (1 + j)\delta/1.65$, where δ is of the order of 1 m.

Consequently, it is necessary to determine the function $\theta(x, y)$ from Equation (1.8), subject to the boundary condition (1.11) and conditions (1.2) and (1.4). In view of (1.7), the latter become

$$\theta = A \quad \text{at} \quad y = 0 \; |x| > l, \qquad \theta = B \quad \text{at} \quad y = -H \tag{1.13}$$

The problem can be simplified somewhat, if we put

$$\theta(x, y) = \theta_1(x, y) + U(x, y)$$

$$\theta_1 = \left(\frac{A}{\sinh \mu_2} - B \coth \mu_2\right) \sinh \mu_2 \left(1 + \frac{y}{H}\right) + B \cosh \mu_2 \left(1 + \frac{y}{H}\right) \tag{1.14}$$

$$(\mu_2 = (1 + t)\, \sigma_2 H)$$

where the function $U(x, y)$ is determined by Equation (1.8) subject to (1.11) and (1.13) and satisfies the conditions

$$\lambda_2 \frac{\partial U}{\partial y} + k_1 U = v_c^\circ \quad \text{at} \quad y = 0 \; |x| < l \tag{1.15}$$

$$\left(v_c^\circ = k_2 C - A\left(k_1 + \frac{\lambda_2 \mu_2}{H} \coth \mu_2\right) + B \frac{\lambda_2 \mu_2}{H \sinh \mu_2}\right)$$

$$U = 0 \quad \text{at} \quad y = 0, \; |x| > l \text{ and at } y = -H \tag{1.16}$$

We note here that in the case of a multi-chamber cooler provided with an external layer of insulation in the ground around the outer walls of the cooler, the quantity C constitutes a piecewise continuous function equal to A at the sections of the external insulation.

2. Construction of solution. We shall prove that the function $U(x, y)$ determined by Equation (1.8) is regular in the strip $-H < y < 0$. Furthermore, it is unique if it satisfies conditions (1.15) and (1.16) for given v_c^0. Indeed, let $U_1(x, y)$ and $U_2(x, y)$ denote two functions, both regular, in the strip $-H < y < 0$ and satisfying Equation (1.8) as well as the conditions (1.15) and (1.16); then, the function $U_0 = U_1 - U_2$ satisfies the conditions

$$\frac{\partial U_0}{\partial y} = -\frac{k}{\lambda_2} U_0 \quad \text{at } y = 0, \ |x| < l, \qquad U_0 = 0 \ \text{at } y = -H \ \text{and} \ y = 0, \ |x| > l$$

and vanishes at $x \to \pm\infty$. Applying Green's formula, we obtain

$$\iint_S \left(\nabla U_0^* \cdot \nabla U_0 + \frac{j\omega}{a_2} U_0^* U_0\right) dS = -\frac{k_1}{\lambda_2} \int_{-l}^{l} U_0^* U_0 dx \tag{2.1}$$

where S denotes an area in the strip $-H < y < 0$ and U_0^* is complex-conjugate. From (2.1) we obtain

$$\iint_S \nabla U_0^* \cdot \nabla U_0 dS = -\frac{\operatorname{Re}(k_1)}{\lambda_2} \int_{-l}^{l} |U_0|^2 dx \qquad -\frac{\omega}{a_2} \iint_S |U_0|^2 dS = -\frac{\operatorname{Im}(k_1)}{\lambda_2} \int_{-l}^{l} |U_0|^2 dx \tag{2.2}$$

In order to calculate the signs of the real and imaginary parts of the coefficient k_1, we shall make use of simplified forms of the generalized coefficients of heat transfer. We note that for an insulation made of glass wool $\lambda_u = 0.03$ kcal/m^2 hr °C, $a_u = 10^{-3}$ m^2/hr and a_c is of order 10 kcal / m^2 hr °C. Hence, in the case of annual fluctuations $|\lambda_u \mu_u / a_c \delta| = 0.004$. Consequently, and with an accuracy which is sufficient for practical applications, it is possible to neglect this ratio in Equations (1.12) and to simplify*

$$k_1 = k\mu_u \coth \mu_u = \frac{kz\,[\sinh 2z + \sin 2z + j(\sinh 2z - \sin 2z)]}{2(\cosh^2 z - \cos^2 z)}$$

$$k_2 = \frac{k\mu_u}{\sinh \mu_u} = \frac{kz\,[\sinh z \cos z + \cosh z \sin z + j(\sinh z \cos z - \cosh z \sin z)]}{\cosh^2 z - \cos^2 z} \tag{2.3}$$

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* For $|\mu_u| \gg 1$ we obtain the limiting values $k_1 = \lambda_u(1 + j)\,\sigma_u$ and $k_2 = 0$ which reduce the boundary conditions (1.11) to those studied by Leontovich.

$$k = \frac{\lambda_u}{\delta} \approx \left(\frac{1}{a_c} + \frac{\delta}{\lambda_u}\right)^{-1}, \qquad z = \sigma_u \delta$$

where k denotes the approximate value of the steady-state coefficient of heat transfer. It follows from (2.3) that $\mathrm{Re}(k_1) > 0$ and $\mathrm{Im}(k_1) > 0$, and according to (2.2) we have $U_0 = 0$. We begin by writing down the exact solution for the case when underground water is absent ($H = \infty$, $\theta_1 = A \exp[(1 + j)\sigma_s y]$). In order to do that we transform to an elliptic system of coordinates ξ and η with the aid of the relations

$$x = l \cosh\xi \cos\eta, \qquad y = l \sinh\xi \sin\eta \tag{2.4}$$

Conditions (1.15) and (1.16) now assume the form

$$\frac{\partial u}{\partial \xi} + \nu_1 \sin\eta u = v_c \sin\eta \quad \text{at} \quad \xi = 0, \ -\pi < \eta < 0,$$

$$u = 0 \quad \text{at} \quad \eta = 0, \ \eta = -\pi \qquad (\xi > 0)$$

$$\left(v_c = \nu_2 C - (\nu_1 + \mu) A, \ \nu_1 = \frac{k_1 l}{\lambda_g}, \ \nu_2 = \frac{k_2 l}{\lambda_g}, \ \mu = (1 + j)\sigma_g l\right) \tag{2.5}$$

and Equation (1.8) transforms to

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} - \mu^2 (\cosh^2\xi - \cos^2\eta) u = 0 \qquad \left(\mu^2 = j\frac{\omega l^2}{a_g}\right) \tag{2.6}$$

We make use of the completeness of the derivatives of the Mathieu function $\mathrm{Se}_n(\xi)\mathrm{se}_n(\eta)$, which constitute particular solutions of Equation (2.6), where $\mathrm{se}_n(\eta)$ constitute orthogonal systems of periodic functions and which are normalized as follows:

$$\int_{-\pi}^{\pi} [\mathrm{se}_n(\eta)]^2 d\eta = \pi \qquad (n = 1, 2, \ldots)$$

The function $\mathrm{se}_n(\eta)$ can be represented by the Fourier series [2]

$$\mathrm{se}_n(\eta) = \sum_{r=1}^{\infty}{}' B_{nr} \sin r\eta \tag{2.7}$$

where the indices n and r are both even or odd and the coefficients B_{nr} are entire functions of the parameter $q = -1/4\,\mu^2$. The function $\mathrm{Se}_n(\xi)$ can also be represented in the form of a series of Bessel functions [3,4].

We shall represent the function U in the form of the series

$$U = \sum_{n=1}^{\infty} a_n \frac{\mathrm{Se}_n(\xi)}{\mathrm{Se}_n(0)} \mathrm{se}_n(\eta) \tag{2.8}$$

This expression satisfies the second condition (2.5). In view of the properties of the function $\mathrm{Se}_n(\xi)$, the series assumes the damping of the temperature waves as $y \to -\infty$. In order to determine the coefficients a_n, we substitute (2.8) into the first condition (2.5), when we obtain

$$\sum_{n=1}^{\infty} \frac{\mathrm{Se}_n'(0)}{\mathrm{Se}_n(0)} a_n \mathrm{se}_n(\eta) + \nu_1 \sum_{m=1}^{\infty} a_m \sin\eta\, \mathrm{se}_m(\eta) = v_c \sin\eta$$
$$\left(-\pi < \eta < 0,\ \mathrm{Se}_n'(0) = \left(\frac{d\,\mathrm{Se}_n}{d\xi}\right)_{\xi=0}\right) \tag{2.9}$$

We now expand the following functions in terms of the Mathieu function $\mathrm{se}_n(\eta)$ in the interval $-\pi < \eta < 0$. We obtain

$$\sin\eta\, \mathrm{se}_m(\eta) = \sum_{n=1}^{\infty} d_{nm} \mathrm{se}_n(\eta) \qquad \left(d_{nm} = \frac{2}{\pi} \int_{-\pi}^{0} \sin\eta\, \mathrm{se}_m(\eta)\, \mathrm{se}_n(\eta)\, d\eta\right) \tag{2.10}$$

$$v_c \sin\eta = \sum_{n=1}^{\infty} b_n \mathrm{se}_n(\eta) \qquad \left(b_n = \frac{2}{\pi} \int_{-\pi}^{0} v_c \sin\eta\, \mathrm{se}_n(\eta)\, d\eta\right) \tag{2.11}$$

Evaluating the coefficients d_{nm} with the aid of (2.7), we obtain

$$d_{nm} = \sum_{r,i=1}^{\infty} B_{nr} B_{mi} l_{ri} \tag{2.12}$$

where the coefficients l_{nm} are different from zero only if r and i are both even or odd and are of the form

$$l_{2p+1,\,2q+1} = -\frac{2}{\pi}\left[\frac{1}{4(p+q+1)^2-1} - \frac{1}{4(p-q)^2-1}\right]$$
$$l_{2p,\,2q} = -\frac{2}{\pi}\left[\frac{1}{4(p+q)^2-1} - \frac{1}{4(p-q)^2-1}\right] \tag{2.13}$$

Consequently, the coefficients d_{nm} are also different from zero when n and m are both even or odd.

Substituting (2.10) and (2.11) into (2.9) and equating the coefficients of $\mathrm{se}_n(\eta)$, we obtain two independent systems of equations for the coefficients a_n:

$$\frac{\mathrm{Se}_{2s+1}'(0)}{\mathrm{Se}_{2s+1}(0)} a_{2s+1} + \nu_1 \sum_{l=0}^{\infty} d_{2s+1,\,2l+1} a_{2l+1} = b_{2s+1} \qquad (s = 0, 1, \ldots) \tag{2.14}$$

$$\frac{\mathrm{Se}_{2s}'(0)}{\mathrm{Se}_{2s}(0)}\, a_{2s} + \nu_1 \sum_{l=1}^{\infty} d_{2s,\,2l}\, a_{2l} = b_{2s} \qquad (s = 1, 2, \ldots) \tag{2.15}$$

The coefficients a_n can now be determined with the aid of infinite determinants [5], and it is noted that in view of the uniqueness of the solution these determinants are different from zero. In the case which is important in practical applications, $v_c(x)$ is an even function, and hence all $b_{2s} = 0$ and, consequently, $a_{2s} = 0$. We remark in particular that for v_c = const we have $b_{2s+1} = v_c B_{2s+1}$.

When $|\mu^2|$ is small, the function $\mathrm{Se}_n(\xi) \approx \exp(-n\xi)$ and $d_{nm} = l_{nm}$. For this reason, when $|\mu^2|$ is small, the systems of equations (2.14) and (2.15) become identical with those discussed in [1], i.e. with systems whose solvability has been demonstrated.

Having determined the coefficients a_n, it is possible to obtain the non-steady distribution of temperature under the insulation of the cooler by employing Formulas (1.7), (1.14) and (2.8). In particular, the temperature fluctuations at the center under the insulation can be determined by ($v_c(x)$ is an even function)

$$\theta(0,0,t) = \left[A + \sum_{s=0}^{\infty} a_{2s+1}\, \mathrm{se}_{2s+1}\left(-\frac{\pi}{2}\right)\right] e^{j\omega t}$$
$$\left(\mathrm{se}_{2s+1}\left(-\frac{\pi}{2}\right) = -\sum_{p=0}^{\infty} (-1)^p B_{2s+1,\,2p+1}\right) \tag{2.16}$$

It is now easy to calculate the quantity of heat Q transferred to the cooler per unit time. In order to do this, we use the equation

$$Q = -\lambda_2 \int^{l} \left(\frac{\partial\theta}{\partial y}\right)_{y=0} dx = \left[-2\lambda_2 \mu A + \lambda_2 \int_{-\pi}^{0} \left(\frac{\partial U}{\partial \xi}\right)_{\xi=0} d\eta\right] e^{j\omega t}$$

In view of (2.7) and (2.8) this becomes

$$Q = -2\lambda_2 \left[\mu A + \sum_{s,\,p=1}^{\infty} \frac{\mathrm{Se}_{2s+1}'(0)}{(2p+1)\,\mathrm{Se}_{2s+1}(0)}\, a_{2s+1} B_{2s+1,\,2p+1}\right] e^{j\omega t} \tag{2.17}$$

The approximate determination of the coefficients a_n can be achieved in the same way, as in the steady case [1].

As a first approximation, it is possible to put $a_2 = \infty$, i.e. $|\mu^2| = 0$. Thus, calculations for $C = 0$ lead to the relation $a_\theta = |\theta(0, 0, t)/A|$ *versus* $\nu = kl/\lambda_2$, represented in Fig. 1 for different values of $z = \sigma_u \delta$.

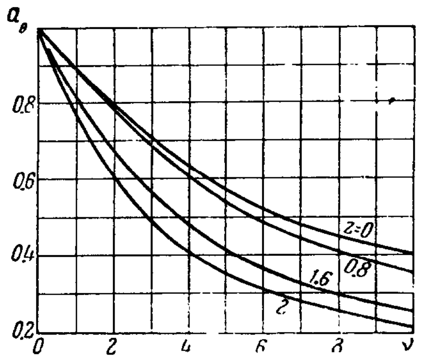

FIG. 1.

If the width of the insulation is sufficiently large, it is possible to obtain a closed solution for the edge effect of penetration of thermal waves under the insulation. In order to achieve this we consider a semi-infinite layer of insulation in the range $(0, \infty)$. For this case the function $u(x, y)$ will be represented in the form of an integral of plane thermal waves $(H = \infty)$:

$$U(x, y) = \int_{-\infty}^{\infty} \Gamma(w)\, e^{-jwx + y\sqrt{w^2+\mu_0^2}}\, dw \qquad (\mu_0 = (1+j)\,\sigma_2) \tag{2.18}$$

The conditions (1.16) and (1.15) lead to the following equations for Γ:

$$\int_{-\infty}^{\infty} \Gamma(w)\, e^{-jwx}\, dw = 0 \quad \text{for } x < 0 \tag{2.19}$$

$$\int_{-\infty}^{\infty} \left(\sqrt{w^2+\mu_0^2} + \mu_0 Z\right) \Gamma(w)\, e^{-jwx}\, dw = v \quad \text{for } x > 0 \tag{2.20}$$

$$\left(Z = \frac{k_1}{\lambda_2 \mu_0}, \; v = \frac{k_2}{\lambda_2} C - (1+Z)\,\mu_0 A \right)$$

The general theory of this type of equation [6-9] is based on the factorization of the expression

$$\sqrt{w^2+\mu_0^2} + \mu_0 Z = f_+(w)\, f_-(w) \tag{2.21}$$

where the function $f_+(w)$ is regular and possesses no zeros in the upper half-plane (Im $w > 0$), and function $f_-(w)$ is regular and possesses no zeros in the lower half-plane (Im $w \leqslant 0$). In addition $f_-(w) = f_+(w)$ and

$$f_{+}(i\mu_0 \cos\tau) = \sqrt{\mu_0(\cos\beta + \cos\tau)}\exp\left(-\frac{1}{2\pi}\int\limits_{\tau-\beta}^{\tau+\beta}\frac{u\,du}{\sin u}\right) \qquad (Z = \sin\beta) \qquad (2.22)$$

We put $v = r\exp(-jsx)$, where s has a negative imaginary part which, in the final solution, must be made to tend to zero. Thus, the solutions of Equations (2.19) and (2.20) can be expressed by the formula

$$\Gamma(w) = -\frac{r}{2\pi j}\frac{1}{f_{-}(s)\,f_{+}(w)\,(w-s)} \qquad (2.23)$$

Formulas (2.18) and (2.23) determine the solution for the form of v under consideration. Subsequent summation leads to the general solution for $v(x)$.

3. Damping properties of the layer of insulation. Let the width of the insulation of the floor of the cooler $2l$ be sufficiently large, so that along the whole width and for $-H < y < 0$ the simplified equation can be used, i.e.

$$\frac{d^2\theta}{dy^2} = (1+j)^2\sigma_2{}^2\,\theta \qquad (3.1)$$

Solving this equation subject to conditions (1.4) and (1.11), we obtain

$$\theta(y,t) \quad \left[D\sinh\mu_2\left(1+\frac{y}{H}\right) + B\cosh\mu_2\left(1+\frac{y}{H}\right)\right]e^{j\omega t} \qquad (3.2)$$

$$D = \frac{k_2 C}{k_1\sinh\mu_2 + \lambda_2\mu_2 H^{-1}\cosh\mu_2} - \frac{B}{k_1\cosh\mu_2 + \lambda_2\mu_2 H^{-1}\sinh\mu_2} \qquad (3.3)$$

In particular, the fluctuations of temperature under the insulation ($y = 0$) are determined by the expression

$$\theta(0,t) = \frac{k_2}{k_1 + \lambda_2\mu_2 H^{-1}\coth\mu_2}Ce^{j\omega t} + \left(\cosh\mu_2 - \frac{\sinh\mu_2}{k_1\cosh\mu_2 + \lambda_2\mu_2 H^{-1}\sinh\mu_2}\right)Be^{j\omega t} \qquad (3.4)$$

We now perform a quantitative analysis of expressions (3.2)-(3.4) for $H = \infty$. In this case, we have

$$\theta(0,t) = \Phi(j\omega)Ce^{j\omega t}, \qquad \Phi(j\omega) = \frac{k_2}{k_1 + (1+j)\sigma_2\lambda_2}$$

$$\theta(y,t) = \theta(0,t)\,e^{(1+j)\sigma_2 y} \qquad (3.5)$$

The last formula shows that the character of the decay of the thermal waves along the depth is the same as in the case of a free, non-insulated ground, first analyzed by Fourier. However, the initial amplitude at $y = 0$ is not equal to C, but is determined by the forcing function $\Phi(j\omega)$ which characterizes the reaction under the insulation in response to a harmonic thermal pulse of unit amplitude. We shall represent this function in the form

$$\Phi(j\omega) = \psi(\omega)\, e^{-j\varepsilon(\omega)}, \qquad \psi(\omega) = |\Phi(j\omega)|$$
$$\varepsilon(\omega) = \arg \Phi^{-1}(j\omega) \tag{3.6}$$

where $\psi(\omega)$ denotes the relative amplitude of temperature fluctuations under the insulation and $\epsilon(\omega)$ is the phase shift.

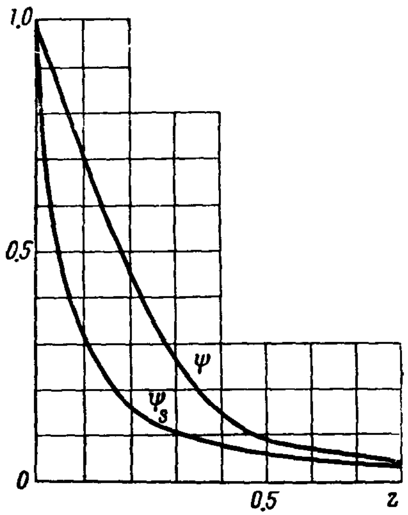

FIG. 2.

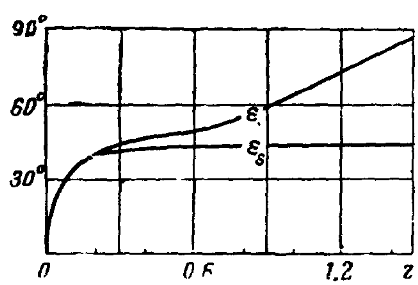

FIG. 3.

For approximate computation we can use the simplified expressions (2.3) from which we obtain

$$\psi(\omega) = \frac{\nu_0}{[\cosh^2 z - \cos^2 z + \nu_0 \sinh 2z + \nu_0^2(\sinh^2 z + \cos^2 z)]^{1/2}}$$
$$\tan\varepsilon(\omega) = \tan z\, \frac{1 + \nu_0 \tanh z}{\nu_0 + \tanh z} \qquad \left(\nu_0 = \frac{\lambda_u}{\lambda_z}\left(\frac{a_z}{a_u}\right)^{1/2},\ z = \sigma_u \delta\right) \tag{3.7}$$

For small values of z, expressions (3.7) reduce to

$$\psi_s(\omega) = \frac{\nu_0}{(\nu_0^2 + 2\nu_0 z + 2z^2)^{1/2}}$$
$$\tan\varepsilon_s(\omega) = \frac{z}{\nu_0 + z} \tag{3.8}$$

where $\psi_s(\omega)$ and $\epsilon_s(\omega)$ can also be obtained by identifying the quantities k_1 and k_2 with the steady-state value of the coefficient of heat transfer k.

The dependence of ψ and ϵ on z is plotted in Figs. 2 and 3 for $\nu_0 = 0.047$, which corresponds to wet soil and glass wool insulation. The graphs show how fast the amplitude of the oscillation decays under the insulation when $z = \sigma_u\delta$ is increased. They show, further, that making use of the steady-state value of the coefficient of heat transfer (ψ_s and ϵ_s) leads to vanishingly small values of amplitude and phase shift.

If the temperature fluctuations $\Delta\theta_c$ are random in nature but steady [10], and if the correlation function $R_c(\tau)$ is known for them, or alternatively, if the spectral distribution $S_c(\omega)$ is known, then the spectral distribution $S_\theta(\omega)$ for the temperature fluctuation under the layer of insulation can be determined from the simple relation

$$S_\theta(\omega) = |\Phi(j\omega)|^2 S_c(\omega) \tag{3.9}$$

In the case of a random process, $R(\tau)$ and $S(\omega)$ are connected by the Fourier transforms [10]

$$\begin{aligned} S(\omega) &= \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau = 2\int_{0}^{\infty} R(\tau)\cos\omega\tau\, d\tau \\ R(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\cos\omega\tau\, d\omega \end{aligned} \tag{3.10}$$

It follows that having determined $R_c(\tau)$ or $S_c(\omega)$ from experimental data, it is possible to calculate $S_\theta(\omega)$ and $R_\theta(\tau)$ from Equations (3.9) and (3.10) and, in particular, the dispersion of the quantity θ equal to $R_\theta(0)$. Having done that, it is not difficult to calculate the remaining statistical characteristics of the distribution of θ. A similar method can be used in the general case too, because Section 2 contains transfer functions for a series of quantities.

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